# 國立中央大學八十五學年度轉學生入學試題卷

數學系 三年級 科目：高等微積分 共 1 頁 第 1 頁

1. (a) Let $f(x) = \frac{1}{x}$, $x \in (0,1]$. Is $f$ continuous on $[0,1]$? Could you redefine $f(0)$ such that $f$ is continuous $[0,1]$? Does $f$ have maximum on $[0,1]$? Prove your results.

   (b) Let $f(x) = x^2 \sin\frac{1}{x}$ for $x \neq 0$; $= 0$ for $x = 0$. Find $\lim_{x\to 0} f(x)$. Is $f$ continuous, uniformly continuous on $[0,1]$? on $(-\infty,\infty)$? Find $f'(0)$, $f''(0)$. (20%)

2. Let $f$ be a continuous mapping of a compact metric space $X$ into a metric space $Y$. Prove: $f(X)$ is compact. (15%)

3. (1) Show by $\varepsilon$-$\delta$ definition of the limit that $\lim_{x\to 1} \frac{1}{x} = 1$

   (2) Show $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$.

   (3) Is $f(x) = x^2$ uniformly continuous on $[0,1]$? on $(-\infty,\infty)$? Verify the results. (15%)

4. Let $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$. (15%)

   (1) Show the series converges uniformly on $[a,\infty)$ for $a > 0$.

   (2) Does the series converges uniformly $(0,\infty)$? Verify the results.

   (3) Is $f(x)$ continuous on $[a,\infty)$ for $a > 0$ and continuous on $(0,\infty)$? Verify your results.

5. If $f$ is a real continuous function on $[a,b]$ and $\alpha$ is an increasing function on $[a,b]$ then show $f$ is Riemann-Stieljes integrable with respect to $\alpha$ over $[a,b]$. (10%)

6. We say $f$ is Lipschitz continuous on $[a,b]$ if $\exists M > 0$ s.t. $|f(x) - f(y)| \leq M|x-y|$ for all $x, y \in [a,b]$. Prove: If $f'$ is continuous on $[a,b]$ then $f$ is Lipschitz continuous on $[a,b]$. (10%)

7. Let $f$, $f_n$ $(n = 1, 2, \cdots)$ be Riemann integrable on $[a,b]$ and $f_n \to f$ on $[a,b]$. Is

$$\int_a^b f\,dx = \lim_{n\to\infty} \int_a^b f_n\,dx? \qquad (*)$$

   If it's not true, give condition on $f_n$ such that $(*)$ is true. Verify your results. (15%)